Fuzzy spheres from inequivalent coherent states quantizations

2007 J. Phys. A: Math. Theor. 40 10225

(http://iopscience.iop.org/1751-8121/40/33/018)

J. Phys. A: Math. Theor. **40** (2007) 10225–10249 doi:10.1088/1751-8113/40/33/018

Fuzzy spheres from inequivalent coherent states quantizations

Jean Pierre Gazeau[1], **Eric Huguet**[1], **Marc Lachièze-Rey**[1] and **Jacques Renaud**[2]

[1] Laboratoire APC, Université Paris 7-Denis Diderot, 10, rue A. Domon et L. Duquet, 75205 Paris Cedex 13, France
[2] Université Paris-Est, Laboratoire APC, UMR CNRS 7164, 10, rue A. Domon et L. Duquet, 75205 Paris Cedex 13, France

E-mail: gazeau@apc.univ-paris7.fr, huguet@apc.univ-paris7.fr, marclr@cea.fr and jacques.renaud@univ-mlv.fr

Received 3 May 2007, in final form 2 July 2007
Published 1 August 2007
Online at stacks.iop.org/JPhysA/40/10225

Abstract
The existence of a family of coherent states (CS) solving the identity in a Hilbert space allows, under certain conditions, to quantize functions defined on the measure space of CS parameters. The application of this procedure to the 2-sphere provides a family of inequivalent CS quantizations based on the spin spherical harmonics (the CS quantization from usual spherical harmonics appears to give a trivial issue for the Cartesian coordinates). We compare these CS quantizations to the usual (Madore) construction of the fuzzy sphere. Due to these differences, our procedure yields new types of fuzzy spheres. Moreover, the general applicability of CS quantization suggests similar constructions of fuzzy versions of a large variety of sets.

PACS numbers: 03.65.Ca, 03.65.Fd, 04.60.Pp, 02.40.Gh

1. Some ideas on quantization

A classical description of a set of data, say X, is usually carried out by considering sets of real or complex functions on X. Depending on the context (data handling, signal analysis, mechanics, etc) the set X will be equipped with a definite structure (topological space, measure space, symplectic manifold, etc) and the set of functions on X which will be considered as *classical observables* must be restricted with regard to the structure on X; for instance, functions viewed as signals should be square integrable with respect to the measure assigned to the set X.

How to provide instead a 'quantum description' of the same set X ? As a first characteristic, the latter replaces—this is a definition—the classical observables by *quantum observables*, which do not commute in general. As usual, these quantum observables will be realized as operators acting on some Hilbert space $\mathcal{H}$, whose projective version will be considered as the set of quantum states. This Hilbert space will be constructed as a subset in the set of functions on X.

The advantage of the coherent states (CS) quantization procedure, in a standard sense (Klauder 1963a, 1963b, Klauder 1995, Berezin 1975) as in recent generalizations (Gazeau *et al* 2003) and applications (Gazeau and Piechocki 2004), is that it requires a minimal significant structure on X, namely the only existence of a measure $\mu(\mathrm{d}x)$, together with a σ-algebra of measurable subsets. As a measure space, X will be given the name of an *observation* set in the present context, and the existence of a measure provides us with a statistical reading of the set of measurable real- or complex-valued functions on X: computing, for instance, average values on subsets with bounded measure. The quantum states will correspond to measurable and square integrable functions on the set X, but not all square integrable functions are eligible as quantum states. The construction of the Hilbert space $\mathcal{H}$ is equivalent to the choice of a class of eligible quantum states, together with a technical condition of continuity. This provides a correspondence between classical and quantum observables by defining a suitable generalization of the standard coherent states.

Generally speaking, a quantization is a procedure which associates with an algebra A of classical observables an algebra $\mathcal{A}$ of quantum observables. The algebra A is usually realized as a commutative Poisson algebra of derivable functions on a symplectic (phase) space. The algebra $\mathcal{A}$ is, however, non-commutative and the quantization procedure must provide a correspondence $A \mapsto \mathcal{A} : f \mapsto F$. Various procedures of quantization exist, which require some of the following conditions.

- With the constant function 1 is associated the unity of $\mathcal{A}$.
- The commutation relations of $\mathcal{A}$ reproduce the Poisson relations of A. Moreover, they offer a realization of the Heisenberg algebra.
- $\mathcal{A}$ is realized as an algebra of operators acting on some Hilbert space.

Most physical quantum theories may be obtained as the result of a canonical quantization procedure. However, the prescriptions for the latter appear quite arbitrary. Moreover, it is difficult, if not impossible, to implement it covariantly. It is thus difficult to generalize this procedure to many systems. Geometric quantization fully exploits the symplectic structure of the phase space, but generally requires more structure, like a symplectic potential, e.g., the Legendre form on the cotangent bundle of a configuration space. In this regard, the deformation quantization appears more general in the sense that it is based on the symplectic structure only and it preserves symmetries (symplectomorphisms).

The coherent state quantization (CSQ) presented here appears more general since it does not even require a symplectic or Poisson structure. The only structure that a space X must possess is a measure. This procedure can be considered from different viewpoints, which are as follows.

- It is mostly genuine in the sense that it verifies all the requirements above, including those relative to the Poisson structure when the later is present. It however appears more general.
- The coherent state quantization may also be seen as a 'fuzzyfication' of X: the algebra A of functions on X is replaced by an algebra $\mathcal{A}$ of operators, which may be seen as the 'coordinates' of a fuzzy version of X.

  It is well known that (some aspects of) ordinary quantum mechanics may be seen as a non-commutative version of the geometry of the phase space, where position and momentum operators do not commute. In this regard, the quantization of a set of data makes a fuzzy (non-commutative) geometry to emerge (Madore 1995). We will show explicitly how the CS quantization of the ordinary sphere leads to a fuzzy geometry.

However, although $\mathcal{A}$ is isomorphic to the algebra of the usual fuzzy sphere, the CSQ does not lead to the usual fuzzy geometry, as we will show explicitly: the correspondence classical $\mapsto$ quantum is different, and the CSQ provides, by construction, an action of $\mathcal{A}$ on a Hilbert space of functions on X. In this sense, the CSQ can really be seen as a different and more complete fuzzyfication procedure.

- Finally, this procedure is, to a certain extent, a change of point of view in considering the system X, not necessarily a path to quantum physics. In this sense, it could be called a discretization or a regularization (Taylor 2001). It shows a similitude with standard procedures pertaining to signal processing, for instance those involving wavelets, which are coherent states for the affine group transforming the half-plane time scale into itself (Daubechies 1992, Ali *et al* 2000). In many respects, the choice of a quantization appears here as the choice of a resolution in looking at the system.

In section 2 we present a construction of coherent states which is very general and encompasses most of the known constructions, and we derive from the existence of a CS family what we call the CS quantization. This quantization extends to various situations the well-known Klauder–Berezin quantization. The formalism is illustrated with the standard Glauber–Klauder–Sudarshan coherent states and the related canonical quantization of the classical phase space of the motion on the real line.

In section 3, we apply the formalism to the sphere S^2 by using orthonormal families of spin spherical harmonics $({}_\sigma Y_{jm})_{-j\leqslant m\leqslant j}$ (Newman and Penrose 1966, Goldberg *et al* 1967, Campbell 1971). For a given σ such that $2\sigma \in \mathbb{Z}$ and j such that $2|\sigma| \leqslant 2j \in \mathbb{N}$, there corresponds a continuous family of coherent states and the subsequent $(2j+1)$-dimensional quantization of the 2-sphere. For a given j, we thus get $2j+1$ *inequivalent* quantizations, corresponding to the possible values of σ. Note that the classical Gilmore–Radcliffe case (Gilmore 1972, Radcliffe 1971) correspond to the particular value $\sigma = j$, and that for a generic σ our coherent states are the $SU(2)$ Perelomov coherent states (Perelomov 1972, Perelomov 1986) built from the UIR group action on a fiducial state $|j,\sigma\rangle$. On the other hand, the case $\sigma = 0$ is proved to be singular in the sense that it leads to a null quantization of the Cartesian coordinates of the 2-sphere.

Section 4 establishes the link between the CS quantization approach to the 2-sphere and the Madore construction (Madore 1995, Lachièze-Rey *et al* 2003) of the fuzzy sphere. We examine in that section the question of equivalence between the two procedures. Note that a construction of the fuzzy sphere based on Gilmore–Perelomov–Radcliffe coherent states (in the case $\sigma = j$) has already been carried out by Grosse and Prešnajder (1993). They proceed to a covariant symbol calculus *à la* Berezin with its corresponding $\star$-product. However, their approach is different from ours.

The appendices give an exhaustive set of formulae, particularly concerning the spin spherical harmonics, needed for a complete description of our CS approach to the 2-sphere.

2. Coherent states

2.1. The construction

The (classical) system to be quantized is considered as a set of data, $X = \{x \in X\}$, assumed to be equipped with a measure μ defined on a σ-field $\mathcal{B}$. We consider the Hilbert spaces $L^2_{\mathbb{K}}(X,\mu)$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) of real or complex functions, with the usual Hermitian inner product $\langle f|g\rangle$. The quantization is defined by the choice of a closed subspace $\mathcal{H}$ of $L^2_{\mathbb{K}}(X, \mathrm{d}\mu)$. The only requirements on $\mathcal{H}$, in addition to be a Hilbert space, amount to the following technical conditions.

- For all $\psi \in \mathcal{H}$ and all x, $\psi(x)$ is well defined (this is, of course, the case whenever X is a topological space and the elements of $\mathcal{H}$ are continuous functions),
- the linear map (evaluation map)

$$\delta_x : \mathcal{H} \to \mathbb{K} \qquad \psi \mapsto \psi(x) \tag{1}$$

is continuous with respect to the topology of $\mathcal{H}$, for almost all x.

The latter condition is realized as soon as the space $\mathcal{H}$ is finite dimensional since all the linear forms are continuous in this case. We see below that some other examples can be found.

As a consequence, using the Riesz theorem, there exists, for almost all x, a unique element $p_x \in \mathcal{H}$ (a function) such that

$$\langle p_x|\psi\rangle = \psi(x). \tag{2}$$

We define the *coherent states* as the normalized vectors corresponding to p_x written in Dirac notation:

$$|x\rangle \equiv \frac{|p_x\rangle}{[\mathcal{N}(x)]^{\frac{1}{2}}}, \qquad \text{where} \quad \mathcal{N}(x) \equiv \langle p_x|p_x\rangle. \tag{3}$$

One can see at once that, for any $\psi \in \mathcal{H}$,

$$\psi(x) = [\mathcal{N}(x)]^{\frac{1}{2}}\langle x|\psi\rangle. \tag{4}$$

As a consequence, one obtains the following resolution of the identity of $\mathcal{H}$ which is at the basis of the whole construction:

$$\mathrm{Id}_{\mathcal{H}} = \int |x\rangle\langle x|\mathcal{N}(x)\mu\,(\mathrm{d}x). \tag{5}$$

This equation is a direct consequence of the following equalities:

$$\begin{aligned}\langle \psi_1| \int |x\rangle\langle x|\mathcal{N}(x)\mu(\mathrm{d}x)|\psi_2\rangle &= \int \langle\psi_1|x\rangle\langle x|\psi_2\rangle\mathcal{N}(x)\mu(\mathrm{d}x)\\ &= \int \psi_1^*(x)\psi_2(x)\mu(\mathrm{d}x)\\ &= \langle\psi_1|\psi_2\rangle,\end{aligned}$$

which hold for any $\psi_1, \psi_2 \in \mathcal{H}$.

Note that

$$\phi(x) = \int_X \sqrt{\mathcal{N}(x)\mathcal{N}(x')}\langle x|x'\rangle\phi(x')\mu(\mathrm{d}x'), \qquad \forall\phi \in \mathcal{H}. \tag{6}$$

Hence, $\mathcal{H}$ is a reproducing Hilbert space with kernel

$$K(x, x') = \sqrt{\mathcal{N}(x)\mathcal{N}(x')}\langle x|x'\rangle, \tag{7}$$

and the latter assumes finite diagonal values (a.e.), $K(x, x) = \mathcal{N}(x)$, by construction. Note that this construction yields an embedding of X into $\mathcal{H}$, and one could interpret $|x\rangle$ as a state localized at x once a notion of localization has been properly defined on X.

In view of (5), the set $\{|x\rangle\}$ is called a *frame* for $\mathcal{H}$. This frame is said to be overcomplete when the vectors $\{|x\rangle\}$ are not linearly independent (Ali *et al* 1993, Ali *et al* 2004).

We define a *classical observable* over X in a loose way as a function $f : X \mapsto \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). As a matter of fact, we will not retain *a priori* the usual requirements on f like to be real valued and smooth with respect to some topology defined on X.

With any such function f we associate the quantum observable over $\mathcal{H}$ through the map:

$$f \mapsto A_f \equiv \int_X \mathcal{N}(x)\mu(\mathrm{d}x) f(x)|x\rangle\langle x|. \tag{8}$$

The operator corresponding to a real function is Hermitian by construction. Hereafter, we will also use the notation $\tilde{f}$ for A_f.

The existence of the continuous frame $\{|x\rangle\}$ enables us to carry out a symbolic calculus in the style of Berezin–Lieb (Berezin 1975, Lieb 1994). With each linear, self-adjoint operator (observable) $\mathcal{O}$ acting on $\mathcal{H}$, one associates the *lower (or covariant) symbol*

$$\check{\mathcal{O}}(x) \equiv \langle x|\,\mathcal{O}\,|x\rangle, \tag{9}$$

and the *upper (or contravariant) symbol* (not necessarily unique) $\widehat{\mathcal{O}}$ such that

$$\mathcal{O} = \int_X \mathcal{N}(x)\mu(\mathrm{d}x)\widehat{\mathcal{O}}(x)|x\rangle\langle x|. \tag{10}$$

Note that f is a upper symbol of A_f.

The technical conditions and the definition of coherent states can be easily expressed when we have a Hilbertian basis of $\mathcal{H}$. Let $(\phi_n)_{n\in I}$ be such a basis; the technical condition is equivalent to

$$\sum_n |\phi_n(x)|^2 < \infty \text{ a.e.} \tag{11}$$

The coherent state is then defined by

$$|x\rangle = \frac{1}{(\mathcal{N}(x))^{\frac{1}{2}}}\sum_n \phi_n^*(x)\phi_n \qquad \text{with} \qquad \mathcal{N}(x) = \sum_n |\phi_n(x)|^2.$$

To a certain extent, the quantization scheme presented here consists in adopting a certain point of view in dealing with X, determined by the choice of the space $\mathcal{H}$. This choice specifies the admissible quantum states, and the correspondence 'classical observables versus quantum observables' follows.

2.2. *The standard coherent states*

Let us illustrate the above construction for the dynamics of a particle moving on the real line. This leads to the well-known Klauder–Glauber–Sudarshan coherent states (Klauder and Skagerstam 1985) and the subsequent so-called *canonical quantization* (with a slight difference of notation). The construction can be easily extended to the dynamics of the particle in a flat higher dimensional *spacetime*. The observation set X is the classical phase space $\mathbb{R}^2 \simeq \mathbb{C} = \left\{z = \frac{1}{\sqrt{2}}(q + \mathrm{i}p)\right\}$ (in complex notations) of a particle with 1 degree of freedom. The symplectic form identifies with $\frac{\mathrm{i}}{2}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} \equiv \mathrm{d}^2 z$, the Lebesgue measure of the plane. Here we adopt the Gaussian measure on X, $\mu(\mathrm{d}z) = \frac{1}{\pi}\,\mathrm{e}^{-|z|^2}\,\mathrm{d}^2 z$.

The quantization of X is hence achieved by a *choice of polarization* (in the language of geometric quantization): the selection, in $L^2(X, \mathrm{d}\mu)$, of the Hilbert subspace $\mathcal{H}$ defined as the so-called Fock–Bargmann space of all antiholomorphic entire functions that are square integrable with respect to the Gaussian measure.

The Hilbertian basis is given by the functions $\phi_n(z) \equiv \frac{\bar{z}^n}{\sqrt{n!}}$, the normalized powers of the conjugate of the complex variable z. Thus, since $\sum_n \frac{|z|^2}{n!} = \mathrm{e}^{|z|^2}$, the coherent states read

$$|z\rangle = \mathrm{e}^{-\frac{|z|^2}{2}}\sum_n \frac{z^n}{\sqrt{n!}}|n\rangle, \tag{12}$$

where $|n\rangle$ stands for ϕ_n, and one easily checks the normalization and resolution of the unity:

$$\langle z|z\rangle = 1, \qquad \frac{1}{\pi}\int_{\mathbb{C}} |z\rangle\langle z| \mathrm{d}^2 z = \mathrm{Id}_{\mathcal{H}}. \tag{13}$$

Note that the reproducing kernel is simply given by $K(z, z') = \mathrm{e}^{z\bar{z}'}$.

A large class of operators acting on $\mathcal{H}$ are yielded by using (8), precisely all operators which can be expressed in a diagonal form with respect the CS family. We thus have for the most basic one,

$$a \equiv A_z = \frac{1}{\pi}\int_{\mathbb{C}} z|z\rangle\langle z|\, \mathrm{d}^2 z = \sum_n \sqrt{n+1}|n\rangle\langle n+1|, \tag{14}$$

which appears as the lowering operator, $a|n\rangle = \sqrt{n}|n-1\rangle$. Its adjoint $a^\dagger$ is obtained by replacing z by $\bar{z}$ in (14), and we get the factorization $N = a^\dagger a$ for the number operator, together with the commutation rule $[a, a^\dagger] = \mathrm{Id}_{\mathcal{H}}$. Also note that $a^\dagger$ and a realize themselves as a multiplication operator and derivation operator respectively when acting on $\mathcal{H}$, $a^\dagger f(z) = zf(z)$, $af = \mathrm{d}f/\mathrm{d}z$. From $q = \frac{1}{\sqrt{2}}(z+\bar{z})$ et $p = \frac{1}{\sqrt{2}i}(z-\bar{z})$, one easily infers by linearity that q and p are upper symbols for $\frac{1}{\sqrt{2}}(a+a^\dagger) \equiv Q$ and $\frac{1}{\sqrt{2}i}(a-a^\dagger) \equiv P$ respectively. In consequence, the (essentially) self-adjoint operators Q and P obey the canonical commutation rule $[Q, P] = \mathrm{i}\mathrm{Id}_{\mathcal{H}}$, and for this reason fully deserve the name of position and momentum operators of the usual (Galilean) quantum mechanics, together with all localization properties specific to the latter.

3. Quantizations of the 2-sphere

3.1. The 2-sphere

We now apply our method to the quantization of the observation set $X = S^2$, the unit 2-sphere. This is not to be confused with the quantization of the phase space for the motion on the 2-sphere (i.e. quantum mechanics on the 2-sphere; see, for instance, (Kowalski and Rembielinski 2000, 2001, Hall and Mitchell 2002)). A point of X is denoted by its spherical coordinates, $x = (\theta, \phi)$. Through the usual embedding in $\mathbb{R}^3$, we may see x as a point $\mathbf{x} = (x^i) \in \mathbb{R}^3$ obeying $\sum_{i=1}^3 (x^i)^2 = 1$. We adopt on S^2 the normalized measure $\mu(\mathrm{d}x) = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi/4\pi$, proportional to the SO(3)-invariant measure, which is also the surface element.

We know that μ is a symplectic form, with the canonical coordinates $q = \phi$, $p = -\cos\theta$. This allows us to see S^2 itself as the phase space for the theory of (classical) angular momentum. In this spirit, we will be able to interpret our procedure as the construction of families of *spin coherent states* including the Gilmore–Radcliffe and analogous to the Perelomov ones (Perelomov 1986) (hereafter, GPR). Also, our construction will take advantage of the group action of SO(3) on S^2 embedded in $\mathbb{R}^3$. This three-dimensional group acts as isometries in $\mathbb{R}^3$, as rotations in S^2. However, we emphasize again that our quantization procedure is based on the only existence of a measure, and may be used in the absence of a metric or symplectic structure.

3.2. The CS quantization of the 2-sphere

3.2.1. The Hilbert space and the coherent states. At the basis of the CS quantization procedure is the choice of a finite dimensional Hilbert space, which is a subspace of $L^2(S^2)$, and which carries a UIR of the group $SU(2)$. We write its dimension $(2j+1)$, with j as the integer or half-integer. Although it could have appeared natural to select this space as V^j, the linear span of ordinary spherical harmonics Y_{jm}, this choice would not allow us to consider half-integer values of j. Moreover, it happens that the quantization so obtained gives trivial results for the Cartesian coordinates. Namely, the quantum counterparts of the Cartesian coordinates (or, equivalently, the spherical harmonics Y_{1m}) are identically zero. Thus, we are led to define $\mathcal{H}$ on a general setting as the linear span of spin spherical harmonics (hereafter SSHs).

3.2.2. The spin spherical harmonics. We define $\mathcal{H} = \mathcal{H}^{\sigma j}$ as the vector space spanned by the spin spherical harmonics ${}_\sigma Y_{j\mu} \in L^2(S^2)$, where $-j \leqslant \sigma, \mu \leqslant j$, and σ is fixed in this range. Note that σ and j are both integers or semi-integers. The SSHs were first introduced in (Newman and Penrose 1966) (see also (Campbell 1971) and (Goldberg *et al* 1967) for their main properties). In view of their importance in the context of the present work, they are comprehensively described in appendix A. The special case $\sigma = 0$ corresponds to the ordinary spherical harmonics

$$ {}_0Y_{jm} = Y_{jm}. $$

A CS quantization is defined after a choice of values for j and σ that we consider as fixed in the following. With the usual inner product of $L^2(S^2)$, the SSHs provide an ON basis $({}_\sigma Y_{j\mu})_{\mu=-j\ldots j}$ of $\mathcal{H}^{\sigma j}$ (hereafter, the SSH basis).

The Hilbert space $\mathcal{H}^{\sigma j}$ carries the $(2j+1)$-dimensional UIR of $SU(2)$ (see appendix A). The generators of $SU(2)$ in this representation can be taken as those corresponding to the three rotations around the orthogonal axes of x^1, x^2, x^3. They are called the 'spin' angular momentum operators (SAMOs, to be distinguished from the usual angular momentum operators J_i), and will be written as $\Lambda_a^{\sigma j}$. Hereafter, the index $a = 1, 2, 3$ will refer to the three spatial directions. We have $\Lambda_a^{0j} = J_a$, the usual angular momentum operators. As usual, we define $\Lambda_\epsilon^{\sigma j} = \Lambda_1^{\sigma j} + \epsilon\,\mathrm{i}\Lambda_2^{\sigma j}$, $\epsilon = \pm 1$. All these generators obey the usual commutation relations of the group $SU(2)$. They act on the ON basis as

$$ \Lambda_3^{\sigma j}\,{}_\sigma Y_{j\mu} = \mu\,{}_\sigma Y_{j\mu}, \qquad \Lambda_\epsilon^{\sigma j}\,{}_\sigma Y_{j\mu} = a_\epsilon(j,\mu)\,{}_\sigma Y_{j\mu+\epsilon}, \tag{15} $$

where $a_\epsilon(j,\mu)$, given in (A.39)–(A.40), are the same as for the usual angular momentum operators J_a.

The SSH basis allows us to identify $\mathcal{H}^{\sigma j}$ with $\mathbb{C}^{2j+1}$:

$$ {}_\sigma Y_{j\mu} \rightsquigarrow |\mu\rangle \hookrightarrow (0,\ldots,0,1,0,\ldots,0)^t \qquad \text{with} \qquad \mu = -j, -j+1, \ldots, j, \tag{16} $$

where 1 is at the position μ and the superscript t denotes the transpose. By construction, we have the Hilbertian orthonormality relations:

$$ \langle\mu|\nu\rangle \equiv \int_X \mu(\mathrm{d}x)\,{}_\sigma Y^*_{j\mu}(x)\,{}_\sigma Y_{j\nu}(x) = \delta_{\mu\nu}. \tag{17} $$

The CS construction presented in section 2.1 leads to the following class of coherent states:

$$ |x\rangle = |\theta,\phi\rangle = \frac{1}{\sqrt{\mathcal{N}(x)}} \sum_{\mu=-j}^{j} {}_\sigma Y^*_{j\mu}(x)|\mu\rangle; \qquad |x\rangle \in \mathcal{H}, \tag{18} $$

with

$$ \mathcal{N}(x) = \sum_{\mu=-j}^{j} |{}_\sigma Y_{j\mu}(x)|^2 = \frac{2j+1}{4\pi}. $$

For $\sigma = \pm j$, they reduce to the *spin coherent states* (Gilmore 1972, Radcliffe 1971, Perelomov 1972, Perelomov 1986).

3.2.3. Operators. We call $\mathcal{O}^{\sigma j} \equiv \mathrm{End}(\mathcal{H}^{\sigma j})$ the space of linear operators (endomorphisms) acting on $\mathcal{H}^{\sigma j}$. This is a complex vector space of dimension $(2j+1)^2$ and an algebra for the natural composition of endomorphisms. The SSH basis allows us to write a linear endomorphism of $\mathcal{H}^{\sigma j}$ (i.e. an element of $\mathcal{O}^{\sigma j}$) in a matrix form. This provides the algebra isomorphism

$$\mathcal{O}^{\sigma j} \rightsquigarrow \mathrm{Mat}_{2j+1},$$

the algebra of complex matrices of order $2j+1$, equipped with the matrix product.

The projector $|x\rangle\langle x|$ is a particular linear endomorphism of $\mathcal{H}^{\sigma j}$, i.e. an element of $\mathcal{O}^{\sigma j}$. Being Hermitian by construction, it may be seen as a Hermitian matrix of order $2j+1$, i.e. an element of $\mathrm{Herm}_{2j+1} \subset \mathrm{Mat}_{2j+1}$. Note that Herm_{2j+1} and Mat_{2j+1} have respective (complex) dimensions $(j+1)(2j+1)$ and $(2j+1)^2$.

We get by the construction resolution of identity and normalization:

$$\int_{S^2} \mu(\mathrm{d}x)\mathcal{N}(x)|x\rangle\langle x| = \mathrm{Id}, \qquad \langle x|x\rangle = 1.$$

3.2.4. Observables. According to the prescription (8), the CS quantization associates with the *classical* observable $f : S^2 \mapsto \mathbb{C}$ the *quantum* observable

$$\begin{aligned}\tilde{f} \equiv A_f &= \int \mu(\mathrm{d}x) f(x)\mathcal{N}(x)|x\rangle\langle x| \\ &= \sum_{\mu,\nu=-j}^{j} \int \mu(\mathrm{d}x) f(x)[{}_\sigma Y_{j\mu}(x)]^*_\sigma Y_{j\nu}(x)|\mu\rangle\langle\nu|.\end{aligned} \tag{19}$$

This operator is an element of $\mathcal{O}^{\sigma j} \sim \mathrm{End}(\mathcal{H}^{\sigma j}) \rightsquigarrow \mathrm{Mat}_{(2j+1)}$. Of course, its existence is submitted to the convergence of (19) in the weak sense as an operator integral. The expression above gives directly its expression as a matrix in the SSH basis, with matrix elements $\tilde{f}_{\mu\nu}$:

$$\tilde{f} = \sum_{\mu,\nu=-j}^{j} \tilde{f}_{\mu\nu}|\mu\rangle\langle\nu| \qquad \text{with} \qquad \tilde{f}_{\mu\nu} = \int \mu(\mathrm{d}x) f(x)_\sigma Y^*_{j\mu}(x)_\sigma Y_{j\nu}(x). \tag{20}$$

When f is real valued, the corresponding matrix belongs to $\mathrm{Herm}_{(2j+1)}$. Also, we have $\widetilde{f^*} = (\tilde{f})^\dagger$ (matrix transconjugate), where we have used the same notation for the operator and the associated matrix.

3.2.5. The usual spherical harmonics as classical observables. A usual spherical harmonics $Y_{\ell m}$ is a particular classical observable and, as such, may be quantized. The quantization procedure associates with $Y_{\ell m}$ the operator $\widetilde{Y_{\ell m}}$. The details of the computation are given in appendix A and the result is given in appendix A.13, equation (A.59). We hence obtain the matrix elements of $\widetilde{Y_{\ell m}}$ in the SSH basis:

$$[\widetilde{Y_{\ell m}}]_{\mu\nu} = (-1)^{\sigma-\mu}(2j+1)\sqrt{\frac{(2\ell+1)}{4\pi}} \begin{pmatrix} j & j & \ell \\ -\mu & \nu & m \end{pmatrix} \begin{pmatrix} j & j & \ell \\ -\sigma & \sigma & 0 \end{pmatrix}, \tag{21}$$

in terms of the $3j$-symbols. This generalizes formula (2.7) of (Freidel and Krasnov 2002). This expression is a real quantity.

Any function f on the 2-sphere with reasonable properties (continuity, integrability ...) may be expanded in spherical harmonics as

$$f = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} Y_{\ell m}, \tag{22}$$

from which follows the corresponding expansion of $\tilde{f}$. However, the $3j$-symbols are non-zero only when a triangular inequality is satisfied. This implies that the expansion is cut at a finite value, giving

$$\tilde{f} = \sum_{\ell=0}^{2j} \sum_{m=-\ell}^{\ell} f_{\ell m} \widetilde{Y_{\ell m}}. \tag{23}$$

This relation means that the $(2j+1)^2$ observables $(\widetilde{Y_{\ell m}})_{\ell \leqslant 2j, -\ell \leqslant m \leqslant \ell}$ provide a second (SH) basis of $\mathcal{O}^{\sigma j}$.

$f_{\ell m}$ are the components of the matrix $\tilde{f} \in \mathcal{O}^{\sigma j}$ in this basis.

3.3. The spin angular momentum operators

3.3.1. Action on functions. The Hilbert space $\mathcal{H}^{\sigma j}$ carries a unitary irreducible representation of the group $SU(2)$ with generators $\Lambda_a^{\sigma j}$ (the SAMOs), which belong to $\mathcal{O}^{\sigma j}$. Their action is given in (A.38)–(A.40). Explicit calculations shown in appendix A (see A.66) give the crucial relations:

$$\widetilde{x^a} = K \Lambda_a^{\sigma j} \qquad \text{with} \qquad K \equiv \frac{\sigma}{j(j+1)}. \tag{24}$$

We see here the peculiarity of the ordinary spherical harmonics ($\sigma = 0$) as an ON basis for the quantization procedure: they would lead to a trivial result for the quantized version of the Cartesian coordinates! On the other hand, the quantization based on the Gilmore–Radcliffe spin coherent states yields the maximal value: $K = 1/(j+1)$. Hereafter, we assume $\sigma \neq 0$.

3.3.2. Action on operators. The $SU(2)$ action on $\mathcal{H}^{\sigma j}$ induces the following canonical (infinitesimal) action on $\mathcal{O}^{\sigma j} = \mathrm{End}(\mathcal{H}^{\sigma j})$:

$$\mathcal{L}_a^{\sigma j} \mapsto \mathcal{L}_a^{\sigma j} A \equiv \left[\Lambda_a^{\sigma j}, A\right] \quad \text{(the commutator)}, \tag{25}$$

here expressed through the generators.

We prove in appendix A (A.72) that $\mathcal{L}_a^{\sigma j} \widetilde{Y_{\ell m}} = \widetilde{J_a Y_{\ell m}}$, from which there results

$$\mathcal{L}_3^{\sigma j} \widetilde{Y_{\ell m}} = m \widetilde{Y_{\ell m}} \qquad \text{and} \qquad (\mathcal{L}^{\sigma j})^2 \widetilde{Y_{\ell m}} = \ell(\ell+1) \widetilde{Y_{\ell m}}.$$

We recall that $(\widetilde{Y_{\ell m}})_{\ell \leqslant 2j}$ form a basis of $\mathcal{O}^{\sigma j}$. The relations above make $\widetilde{Y_{\ell m}}$ appear as the unique (up to a constant) element of $\mathcal{O}^{\sigma j}$ that is a common eigenvector to $\mathcal{L}_3^{\sigma j}$ and $(\mathcal{L}^{\sigma j})^2$, with eigenvalues m and $\ell(\ell+1)$ respectively. This implies by linearity that for all f such that $\widetilde{f}$ makes sense,

$$\mathcal{L}_a^{\sigma j} \widetilde{f} = \widetilde{J_a f} \qquad \text{and} \qquad (\mathcal{L}^{\sigma j})^2 \widetilde{f} = \widetilde{J^2 f}.$$

4. Link with the Madore fuzzy sphere

4.1. The construction of the fuzzy sphere in the style of Madore

Let us first recall a usual construction of the fuzzy sphere (see, for instance, Madore (1995, p 148)) that we slightly modify to make the correspondence with the CS quantization. It starts from the decomposition of any smooth function $f \in C^\infty(S^2)$ in spherical harmonics,

$$f = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} Y_{\ell m}. \tag{26}$$

Let us denote by V^ℓ the $(2\ell+1)$-dimensional vector space generated by $Y_{\ell m}$, at fixed ℓ.

Through the embedding of S^2 in $\mathbb{R}^3$ any function in S^2 can be seen as the restriction of a function on $\mathbb{R}^3$ (that we write with the same notation), and under some mild conditions such functions are generated by the homogeneous polynomials in $\mathbb{R}^3$. This allows us to express (26) in a polynomial form in $\mathbb{R}^3$:

$$f(x) = f_{(0)} + \sum_{(i_1)} f_{(i)} x^i + \cdots + \sum_{(i_1 i_2 \ldots i_\ell)} f_{(i_1 i_2 \ldots i_\ell)} x^{i_1} x^{i_2} \ldots x^{i_\ell} + \cdots, \tag{27}$$

where each sub-sum is restricted to V^ℓ and involves all symmetric combinations of the i_k indices, each one varying from 1 to 3. This gives, for each fixed value of ℓ, $2\ell + 1$ coefficients $f_{(i_1 i_2 \ldots i_\ell)}$ (ℓ fixed), which are those of a symmetric traceless $3 \times 3 \times \cdots \times 3$ (ℓ times) tensor.

The fuzzy sphere with $2j + 1$ cells is usually written as $S_{\mathrm{fuzzy},j}$, with j as an integer or semi-integer. Here our slightly modified procedure leads to a different fuzzy sphere that we write ${}_\sigma S_{\mathrm{fuzzy,j}}$. We detail the steps of its standard definition.

(1) We consider a $(2j + 1)$-dimensional irreducible unitary representation (UIR) of $SU(2)$. The standard construction considers the vector space V^j of dimension $2j + 1$, on which the three generators of $SU(2)$ are expressed as the usual $(2j + 1) \times (2j + 1)$ Hermitian matrices J_a. Here we will make a different choice, namely the three SAMOs Λ_j, which correspond to the choice of the representation space $\mathcal{H}^{\sigma j}$ (instead of V^j in the usual construction). Since they obey the commutation relations of $su(2)$,

$$\left[\Lambda_a^{\sigma j}, \Lambda_b^{\sigma j}\right] = \mathrm{i}\epsilon_{abc} \Lambda_c^{\sigma j}, \tag{28}$$

the usual procedure may be applied. As we have seen, $\mathcal{H}^{\sigma j}$ can be realized as the Hilbert space spanned by the spin spherical harmonics $\{{}_\sigma Y_{j\mu}\}_{\mu=-j\ldots j}$, with the usual inner product. The latter provides the SSH (ON) basis.

Since the standard derivation of all properties of the fuzzy sphere rests only upon the abstract commutation rules (28), nothing but the representation space changes if we adopt the representation space $\mathcal{H}$ instead of V.

(2) The operators $\Lambda_a^{\sigma j}$ belong to $\mathcal{O}^{\sigma j}$ and have a Lie algebra structure through the skew products defined by the commutators. But the *symmetrized* products of operators provide a second algebra structure, that we write $\mathcal{O}^{\sigma j}$, at the basis of the construction of the fuzzy sphere: these symmetrized products of $\Lambda_a^{\sigma j}$, up to power $2j$, generate the algebra $\mathcal{O}^{\sigma j}$ (of dimension $(2j + 1)^2$) of all linear endomorphisms of $\mathcal{H}^{\sigma j}$, exactly like the ordinary J_a's do in the original Madore construction. This is the standard construction of the fuzzy sphere, with J_a and V^j replaced by $\Lambda_a^{\sigma j}$ and $\mathcal{H}^{\sigma j}$ respectively.

(3) The construction of the fuzzy sphere (of radius r) is defined by associating an operator $\hat{f}$ in $\mathcal{O}^{\sigma j}$ with any function f. Explicitly, this is done by first replacing each coordinate x^i by the operator

$$\widehat{x^a} \equiv \kappa \Lambda_a^{\sigma j} \equiv \frac{r \Lambda_a^{\sigma j}}{\sqrt{j(j+1)}}, \tag{29}$$

in the above expansion (27) of f (in the usual construction, this would be J_a instead of $\Lambda_a^{\sigma j}$). Next, we replace in (27) the usual product by the symmetrized product of operators, and we truncate the sum at index $\ell = 2j$. This associates with any function f an operator $\hat{f} \in \mathcal{O}^{\sigma j}$.

(4) The vector space Mat_{2j+1} of $(2j+1) \times (2j+1)$ matrices is linearly generated by a number $(2j+1)^2$ of independent matrices. According to the above construction, a basis of Mat_{2j+1} can be selected as formed by all the products of $\Lambda_a^{\sigma j}$ up to the power $2j + 1$ (which is necessary and sufficient to close the algebra).

(5) The commutative algebra limit is restored by letting j go to the infinity while parameter κ goes to zero, and κj is fixed to $\kappa j = r$.

The geometry of the fuzzy sphere $S_{\text{fuzzy},j}$ is thus constructed after making the choice of the algebra of the matrices of the representation, with their matrix product. It is taken as the algebra of operators, which generalize the functions. The rank $(2j+1)$ of the matrices invites us to view them as acting as endomorphisms in a Hilbert space of dimension $(2j+1)$. This is exactly what allows the coherent states quantization introduced in the previous section.

4.2. Operators

We have defined the action on $\mathcal{O}^{\sigma j}$:

$$\mathcal{L}_a^{\sigma j} A \equiv \left[\Lambda_a^{\sigma j}, A\right].$$

Formula (27) expresses any function f of V^ℓ as the reduction to S^2 of homogeneous polynomials, homogeneous of order ℓ:

$$f = \sum_{\alpha,\beta,\gamma} f_{\alpha,\beta,\gamma} (x^1)^\alpha (x^2)^\beta (x^3)^\gamma; \qquad \alpha+\beta+\gamma = \ell.$$

The action of the ordinary momentum operators J_3 and J^2 is straightforward. Namely,

$$J_3 f = \sum_{\alpha,\beta,\gamma} f_{\alpha,\beta,\gamma} (-i)[\beta (x^1)^{\alpha+1} (x^2)^{\beta-1} (x^3)^\gamma - \alpha (x^1)^{\alpha-1} (x^2)^{\beta+1} (x^3)^\gamma],$$

and similarly for J_1 and J_2.

On the other hand, we have by definition

$$\hat{f} = \sum_{\alpha,\beta,\gamma} f_{\alpha,\beta,\gamma} S((\widehat{x^1})^\alpha (\widehat{x^2})^\beta (\widehat{x^3})^\gamma), \tag{30}$$

where $S(\cdot)$ means symmetrization. Recalling $\widehat{x^a} = \kappa \Lambda_a^{\sigma j}$, and using (28), we apply the operator $\mathcal{L}_3^{\sigma j}$ to this expression:

$$\mathcal{L}_3^{\sigma j} \hat{f} \equiv \left[\Lambda_3^{\sigma j}, \hat{f}\right] = \sum_{\alpha,\beta,\gamma} f_{\alpha,\beta,\gamma} \left[\Lambda_3^{\sigma j}, S\left(\hat{x^1}^\alpha \hat{x^2}^\beta \hat{x^3}^\gamma\right)\right]. \tag{31}$$

We prove in appendix B that the commutator of the symmetrized is the symmetrized of the commutator. Then, using the identity

$$[J, AB\cdots M] = [J, A]B\cdots M + A[J, B]\cdots M + \cdots + AB\cdots[J, M],$$

which results easily (by induction) from $[J, AB] = [J, A]B + A[J, B]$, it follows that

$$\mathcal{L}_3^{\sigma j} \hat{f} \equiv \left[\Lambda_3^{\sigma j}, \hat{f}\right] = \sum_{\alpha,\beta,\gamma} f_{\alpha,\beta,\gamma} \left(\mathrm{i}\alpha \hat{x^1}^{\alpha-1} \hat{x^2}^{\beta+1} \hat{x^3}^\gamma - \mathrm{i}\beta \hat{x^1}^{\alpha+1} \hat{x^2}^{\beta-1} \hat{x^3}^\gamma\right). \tag{32}$$

We have thus proven

$$\mathcal{L}_3^{\sigma j} \hat{f} = \widehat{J_3 f}.$$

Similar identities hold for $\mathcal{L}_1^{\sigma j}$, $\mathcal{L}_2^{\sigma j}$ and thus for $(\mathcal{L}^{\sigma j})^2$.

There results that $\widehat{Y_{\ell m}}$ appears as an element of $\mathcal{O}^{\sigma j}$ which is a common eigenvector of $\mathcal{L}_3^{\sigma j}$, with the value m, and of $(\mathcal{L}^{\sigma j})^2$, with the value $\ell(\ell+1)$. Since we have proved above that such an element is unique (up to a constant), there results that each $\widehat{Y_{\ell m}} \propto \widetilde{Y_{\ell m}}$. Thus, $\widehat{Y_{\ell m}}$'s, for $\ell \leqslant j, -j \leqslant m \leqslant j$ form a basis of $\mathcal{A}^j$.

Then the Wigner–Eckart theorem (see appendix A.15) implies that $\widetilde{Y_{\ell m}} = C(\ell)\widehat{Y_{\ell m}}$, where the proportionality constant $C(\ell)$ does not depend on m (that can also be checked directly).

Table 1. Coherent state quantization of the sphere is compared to the standard construction of the fuzzy sphere through the correspondence formula.

	Coherent states fuzzy sphere	Madore-like fuzzy sphere
Hilbert space	$\mathcal{H} = \mathcal{H}^{\sigma j} = \mathrm{span}({}_\sigma Y_{j\mu}) \subset L^2(S^2)$	
Endomorphisms	$\mathcal{O} = \mathcal{O}^{\sigma j} = \mathrm{End}\mathcal{H}^{\sigma j}$	
Spin angular momentum operators	$\Lambda_a^{\sigma j} \in \mathcal{O}$	
Observables	$\widetilde{f} \in \mathcal{O}^{\sigma j}$; $\widetilde{x^a} = K\Lambda_a^{\sigma j}$	$\widehat{f} \in \mathcal{O}^{\sigma j}$; $\widehat{x^a} = \kappa\Lambda_a^{\sigma j}$
Action of angular momentum	$\mathcal{L}_a^{\sigma j}\widetilde{f} \equiv [\Lambda_a^{\sigma j}, \widetilde{f}] = \widetilde{J_a f}$	$\mathcal{L}_a^{\sigma j}\widehat{f} \equiv [\Lambda_a^{\sigma j}, \widehat{f}] = \widehat{J_a f}$
Correspondence	$\widetilde{Y_{\ell m}} = C(\ell)\widehat{Y_{\ell m}}$	

These coefficients can be calculated directly, after remarking that

$$\widehat{Y_{\ell\ell}} \propto (\Lambda_+)^\ell \propto (\widehat{x^1} + \mathrm{i}\widehat{x^2})^\ell.$$

In fact,

$$\widehat{Y_{\ell\ell}} = a(\ell)(\widehat{x^1} + \mathrm{i}\widehat{x^2})^\ell, \qquad a(\ell) = \frac{\sqrt{(2\ell+1)!}}{2^{\ell+1}\sqrt{\pi}\,\ell!}.$$

We obtain

$$C(\ell) = 2^\ell \frac{(-1)^{j+\sigma-2\ell}(2j+1)}{\kappa^\ell}\sqrt{\frac{(2j-\ell)!}{(2j+\ell+1)!}}\begin{pmatrix} j & j & \ell \\ -\sigma & \sigma & 0\end{pmatrix}.$$

5. Discussion

We thus have two families of quantization of the sphere.

- The usual construction of the fuzzy sphere, which depends on the parameter j. This parameter defines the 'size' of the discrete cell.
- The present construction coherent states which make use of coherent states and which depend on two parameters, j and $\sigma \neq 0$.

These two quantizations may be formulated as involving the same algebra of operators (quantum observables) $\mathcal{O}$, acting on the same Hilbert space $\mathcal{H}$ (see table 1). Note that $\mathcal{H}$ and $\mathcal{O}$ are not the Hilbert space and algebra appearing in the usual construction of the fuzzy sphere (when we consider them as embedded in the space of functions on the sphere and of operators acting on them), but they are isomorphic to them, and nothing is changed.

The difference lies in the fact that the quantum counterparts, $\tilde{f}$ and $\hat{f}$, of a given classical observable f differ in both approaches. Thus, the CS quantization really differs from the usual fuzzy sphere quantization. This raises the question if whether the CS quantization is or not a construction of a new type of fuzzy sphere. There results from the calculations above that all properties of the usual fuzzy sphere are shared by the CS quantized version. The only point to be checked is whether it gives the sphere manifold in some classical limit. The answer is positive as far as the classical limit is correctly defined. Simple calculations show that it is obtained as the limit $j \mapsto \infty, \sigma \mapsto \infty$, provided that the ratio σ/j tends to a finite value. Thus, one may consider that the CS quantization leads to a one (discrete) parameter family of fuzzy spheres if we impose relations of the type $\sigma = j - \sigma_0$, for fixed $\sigma_0 > 0$ (for instance).

6. Conclusion

We have first described a general quantization procedure which applies to any measurable set X. It proceeds from the choice of a Hilbert space $\mathcal{H}$ of prescribed dimension. We have presented in details an implementation of this procedure (not necessarily unique) by using an explicit family of coherent states, which realizes a natural embedding of X into $\mathcal{H}$. Actually, we proceed to a 'non-commutative' reading of a given geometry and not of a given dynamical system, and the procedure can be viewed as a (canonical or not) quantization when we restrict it to the latter. This means that we do not consider from the very beginning any time parameter and related evolution.

We have applied this CS procedure to the sphere S^2. We started from a natural basis linked to the UIRs of the group $SU(2)$: for any value of j and σ, we chose the Hilbert space $\mathcal{H}^{\sigma j}$, which carries a UIR of $SU(2)$. Our CS construction associates, with any classical observable $f \in L^2$, a quantum observable $\widetilde{f}$, which belongs to the algebra of endomorphisms $\mathcal{O}^{\sigma j} \equiv \mathrm{End}(\mathcal{H}^{\sigma j})$. On the other hand, we also followed the usual fuzzy sphere construction (with $2j+1$ cells), by replacing the coordinates by operators acting on the same Hilbert space. This allowed us to associate a fuzzy observable $\widehat{f}$ to any classical observable f. Those $\widehat{f}$ form the algebra of operators acting on the fuzzy sphere.

For the particular classical observables provided by the ordinary spherical harmonics, we have shown that the CS quantum observable and the fuzzy observable coincide up to a constant, $\widehat{Y_{\ell m}} = C(\ell)\widetilde{Y_{\ell m}}$, and the explicit value of this constant has been given. However, in general, $\widetilde{f}$ differs from $\widehat{f}$, although the correspondence is easily established from the relation above, through a development in the usual spherical harmonics.

Thus, the CS quantization procedure really differs from the construction of the usual fuzzy sphere. Although they share the same algebra of quantum observables, acting on the same Hilbert space, the CS quantum observables, $\widetilde{f}$, and the fuzzy one, $\widehat{f}$, associated with the same classical observable f differ. There is no way to make them coincide, since the CS quantization with $\sigma = 0$ leads to trivial results.

Our discussion in (5) allows us to consider our CS quantization procedure as a construction of a new family of fuzzy spheres, with properties differing from the standard one. They share most of the properties of the usual fuzzy sphere, but the construction is by far more economic in the sense that

- it does not require a group action on the space to be quantized,
- it does not require an initial expansion of the functions into spherical harmonics.

Applications of procedures of this type to the sphere have appeared in different contexts. For instance, a similar procedure is carried out in Taylor (2001) in order to achieve a *regularization* of a membrane, with the surface S^2, by a mapping of functions to matrices, similar to the one presented here. Despite analog mathematics, the procedure there is not seen as a *quantization* and, according to the author, the regularized theory still requires a further quantization. Similar regularization exists for surfaces of arbitrary genus, and it would be interesting to apply the CS procedure in these cases. Also, it should not be difficult to explore cases with more dimensions, and in particular S^3. This offers possibilities to construct new fuzzy versions of these spaces. Moreover, the authors in (Freidel and Krasnov 2002) have given a description of the fuzzy sphere in terms of $SU(2)$ spin networks. Since the latter plays an important role in the canonical quantization of *general relativity*, this suggests that the application of the CS procedure to the quantization of gravity or to various geometries, compact or non-compact (Gazeau *et al* 2006), could be fruitful, a program that we start to explore. Furthermore, the universality of the CS procedure would allow explicit constructions of spin networks associated with different groups, in particular $SU(3)$. Since it has been claimed that the latter could be of importance for quantum gravity, this reveals to be a promising field of research also.

Appendix A. Spin spherical harmonics

A.1. $SU(2)$-parameterization

$$SU(2) \ni \xi = \begin{pmatrix} \xi_0 + \mathrm{i}\xi_3 & -\xi_2 + \mathrm{i}\xi_1 \\ \xi_2 + \mathrm{i}\xi_1 & \xi_0 - \mathrm{i}\xi_3 \end{pmatrix}. \tag{A.1}$$

In bicomplex angular coordinates,

$$\xi_0 + \mathrm{i}\xi_3 = \cos\omega \mathrm{e}^{\mathrm{i}\psi_1}, \qquad \xi_1 + \mathrm{i}\xi_2 = \sin\omega \mathrm{e}^{\mathrm{i}\psi_2} \tag{A.2}$$

$$0 \leqslant \omega \leqslant \frac{\pi}{2}, \qquad 0 \leqslant \psi_1, \qquad \psi_2 < 2\pi, \tag{A.3}$$

and so

$$SU(2) \ni \xi = \begin{pmatrix} \cos\omega \mathrm{e}^{\mathrm{i}\psi_1} & \mathrm{i}\sin\omega \mathrm{e}^{\mathrm{i}\psi_2} \\ \mathrm{i}\sin\omega \mathrm{e}^{-\mathrm{i}\psi_2} & \cos\omega \mathrm{e}^{-\mathrm{i}\psi_1} \end{pmatrix}, \tag{A.4}$$

in agreement with (Talman 1968).

A.2. Matrix elements of $SU(2)$-UIR

$$D^j_{m_1 m_2}(\xi) = (-1)^{m_1 - m_2}[(j+m_1)!(j-m_1)!(j+m_2)!(j-m_2)!]^{1/2} \times \sum_t \frac{(\xi_0 + \mathrm{i}\xi_3)^{j-m_2-t}}{(j-m_2-t)!} \frac{(\xi_0 - \mathrm{i}\xi_3)^{j+m_1-t}}{(j+m_1-t)!} \frac{(-\xi_2 + \mathrm{i}\xi_1)^{t+m_2-m_1}}{(t+m_2-m_1)!} \frac{(\xi_2 + \mathrm{i}\xi_1)^t}{t!}, \tag{A.5}$$

in agreement with Talman. With angular parameters, the matrix elements of the UIR of $SU(2)$ are given in terms of Jacobi polynomials (Magnus *et al* 1966) by

$$D^j_{m_1 m_2}(\xi) = \mathrm{e}^{-\mathrm{i}m_1(\psi_1+\psi_2)}\, \mathrm{e}^{-\mathrm{i}m_2(\psi_1-\psi_2)}\, i^{m_2-m_1} \sqrt{\frac{(j-m_1)!(j+m_1)!}{(j-m_2)!(j+m_2)!}} \times \frac{1}{2^{m_1}}(1+\cos 2\omega)^{\frac{m_1+m_2}{2}}(1-\cos 2\omega)^{\frac{m_1-m_2}{2}} P^{(m_1-m_2, m_1+m_2)}_{j-m_1}(\cos 2\omega), \tag{A.6}$$

in agreement with (Edmonds 1968) (up to an irrelevant phase factor).

A.3. Orthogonality relations and $3j$-symbols

Let us equip the $SU(2)$ group with its Haar measure:

$$\mu(\mathrm{d}\xi) = \sin 2\omega \,\mathrm{d}\omega \,\mathrm{d}\psi_1 \,\mathrm{d}\psi_2, \tag{A.7}$$

in terms of the bicomplex angular parametrization. Note that the volume of $SU(2)$ with this choice of normalization is $8\pi^2$. The orthogonality relations satisfied by the matrix elements $D^j_{m_1 m_2}(\xi)$ read as

$$\int_{SU(2)} D^j_{m_1 m_2}(\xi) \big(D^{j'}_{m_1' m_2'}(\xi)\big)^* \mu(\mathrm{d}\xi) = \frac{8\pi^2}{2j+1} \delta_{jj'} \delta_{m_1 m_1'} \delta_{m_2 m_2'}. \tag{A.8}$$

In connection with the reduction of the tensor product of two UIRs of $SU(2)$, we have the following equivalent formula involving the so-called $3-j$ symbols (proportional to Clebsch–Gordan coefficients), in the Talman notations:

$$D^{j}_{m_1 m_2}(\xi) D^{j'}_{m'_1 m'_2}(\xi) = \sum_{j'' m''_1 m''_2} (2j''+1) \begin{pmatrix} j & j' & j'' \\ m_1 & m'_1 & m''_1 \end{pmatrix} \begin{pmatrix} j & j' & j'' \\ m_2 & m'_2 & m''_2 \end{pmatrix} \left(D^{j''}_{m''_1 m''_2}(\xi) \right)^*, \quad \text{(A.9)}$$

$$\int_{SU(2)} D^{j}_{m_1 m_2}(\xi) D^{j'}_{m'_1 m'_2}(\xi) D^{j''}_{m''_1 m''_2}(\xi) \mu(\mathrm{d}\xi) = 8\pi^2 \begin{pmatrix} j & j' & j'' \\ m_1 & m'_1 & m''_1 \end{pmatrix} \begin{pmatrix} j & j' & j'' \\ m_2 & m'_2 & m''_2 \end{pmatrix}. \quad \text{(A.10)}$$

One of the multiple expressions of the $3-j$ symbols (in the convention that they are all real) is given by

$$\begin{pmatrix} j & j' & j'' \\ m & m' & m'' \end{pmatrix} = (-1)^{j-j'-m''} \left[\frac{(j+j'-j'')!(j-j'+j'')!(-j+j'+j'')!}{(j+j'+j''+1)!} \right]^{1/2} \sum_s (-1)^s$$
$$\times \frac{[(j+m)!(j-m)!(j'+m')!(j'-m')!(j''+m'')!(j''-m'')!]^{1/2}}{s!(j'+m'-s)!(j-m-s)!(j''-j'+m+s)!(j''-j-m'+s)!(j+j'-j''-s)!}. \quad \text{(A.11)}$$

A.4. Spin spherical harmonics

The spin spherical harmonics, as functions on the 2-sphere S^2, are defined as follows:

$${}_\sigma Y_{j\mu}(\hat{\mathbf{r}}) = \sqrt{\frac{2j+1}{4\pi}} \left[D^{j}_{\mu\sigma}(\xi(\mathcal{R}_{\hat{\mathbf{r}}})) \right]^* = (-1)^{\mu-\sigma} \sqrt{\frac{2j+1}{4\pi}} D^{j}_{-\mu-\sigma}(\xi(\mathcal{R}_{\hat{\mathbf{r}}})) \quad \text{(A.12)}$$

$$= \sqrt{\frac{2j+1}{4\pi}} D^{j}_{\sigma\mu}(\xi^{\dagger}(\mathcal{R}_{\hat{\mathbf{r}}})), \quad \text{(A.13)}$$

where $\xi(\mathcal{R}_{\hat{\mathbf{r}}})$ is a (non-unique) element of $SU(2)$ which corresponds to the space rotation $\mathcal{R}_{\hat{\mathbf{r}}}$ which brings the unit vector $\widehat{\mathbf{e_3}}$ to the unit vector $\widehat{\mathbf{r}}$ with polar coordinates:

$$\widehat{\mathbf{r}} = \begin{cases} x^1 = \sin\theta\cos\phi, \\ x^2 = \sin\theta\sin\phi, \\ x^3 = \cos\theta. \end{cases} \quad \text{(A.14)}$$

We immediately infer from definition (A.12) the following properties:

$$({}_\sigma Y_{j\mu}(\hat{\mathbf{r}}))^\star = (-1)^{\sigma-\mu} {}_{-\sigma}Y_{j-\mu}(\hat{\mathbf{r}}), \quad \text{(A.15)}$$

$$\sum_{\mu=-j}^{\mu=j} |{}_\sigma Y_{j\mu}(\hat{\mathbf{r}})|^2 = \frac{2j+1}{4\pi}. \quad \text{(A.16)}$$

Let us recall here the correspondence (homomorphism) $\xi = \xi(\mathcal{R}) \in SU(2) \leftrightarrow \mathcal{R} \in SO(3) \simeq SU(2)/\mathbb{Z}_2$:

$$\widehat{\mathbf{r}}' = (x'_1, x'_2, x'_3) = \mathcal{R} \cdot \widehat{\mathbf{r}} \longleftrightarrow \quad \text{(A.17)}$$

$$\begin{pmatrix} \mathrm{i}x'_3 & -x'_2 + \mathrm{i}x'_1 \\ x'_2 + \mathrm{i}x'_1 & -\mathrm{i}x'_3 \end{pmatrix} = \xi \begin{pmatrix} \mathrm{i}x_3 & -x_2 + \mathrm{i}x_1 \\ x_2 + \mathrm{i}x_1 & -\mathrm{i}x_3 \end{pmatrix} \xi^{\dagger}. \quad \text{(A.18)}$$

In the particular case of (A.12), the angular coordinates ω, ψ_1, ψ_2 of the $SU(2)$-element $\xi(\mathcal{R}_{\hat{\mathbf{r}}})$ are constrained by

$$\cos 2\omega = \cos\theta, \qquad \sin 2\omega = \sin\theta \quad \text{so} \quad 2\omega = \theta, \tag{A.19}$$

$$\mathrm{e}^{\mathrm{i}(\psi_1+\psi_2)} = \mathrm{i}\mathrm{e}^{\mathrm{i}\phi} \quad \text{so} \quad \psi_1 + \psi_2 = \phi + \frac{\pi}{2}. \tag{A.20}$$

Here we should pay a special attention to the range of values for the angle ϕ, depending on whether j and consequently σ and m are half-integers or not. If j is half-integer, then the angle ϕ should be defined as $\mathrm{mod}(4\pi)$ whereas if j is an integer, it should be defined as $\mathrm{mod}(2\pi)$.

We still have one degree of freedom concerning the pair of angles ψ_1, ψ_2. We leave open the option concerning the σ-dependent phase factor by putting

$$\mathrm{i}^{-\sigma}\,\mathrm{e}^{\mathrm{i}\sigma(\psi_1-\psi_2)} \stackrel{\text{def}}{=} \mathrm{e}^{\mathrm{i}\sigma\psi}, \tag{A.21}$$

where ψ is arbitrary. With this choice and considering (A.5), we get the expression of the spin spherical harmonics in terms of ϕ, $\theta/2$ and ψ:

$$\begin{aligned} {}_\sigma Y_{j\mu}(\hat{\mathbf{r}}) &= (-1)^\sigma\,\mathrm{e}^{\mathrm{i}\sigma\psi}\,\mathrm{e}^{\mathrm{i}\mu\phi}\sqrt{\frac{2j+1}{4\pi}}\sqrt{\frac{(j+\mu)!(j-\mu)!}{(j+\sigma)!(j-\sigma)!}} \\ &\quad\times\left(\cos\frac{\theta}{2}\right)^{2j}\sum_t(-1)^t\binom{j-\sigma}{t}\binom{j+\sigma}{t+\sigma-\mu}\left(\tan\frac{\theta}{2}\right)^{2t+\sigma-\mu}, \end{aligned} \tag{A.22}$$

$$\begin{aligned} &= (-1)^\sigma\,\mathrm{e}^{\mathrm{i}\sigma\psi}\,\mathrm{e}^{\mathrm{i}\mu\phi}\sqrt{\frac{2j+1}{4\pi}}\sqrt{\frac{(j+\mu)!(j-\mu)!}{(j+\sigma)!(j-\sigma)!}} \\ &\quad\times\left(\sin\frac{\theta}{2}\right)^{2j}\sum_t(-1)^{j-t+\mu-\sigma}\binom{j-\sigma}{t-\mu}\binom{j+\sigma}{t+\sigma}\left(\cot\frac{\theta}{2}\right)^{2t+\sigma-\mu}, \end{aligned} \tag{A.23}$$

which are not in agreement with the definitions of Newman and Penrose (1966), Campbell (1971) (note that there is a mistake in the expression given by Campbell, in which $\cos\frac{\theta}{2}$ should read $\cot\frac{\theta}{2}$) and (Hu and White 1997). Besides the presence of different phase factors, the disagreement is certainly due to a different relation between the polar angle θ and the Euler angle.

Now, considering (A.6), we get the expression of the spin spherical harmonics in terms of the Jacobi polynomials, valid in the case in which $\mu \pm \sigma > -1$:

$$\begin{aligned} {}_\sigma Y_{j\mu}(\hat{\mathbf{r}}) &= (-1)^\mu\,\mathrm{e}^{\mathrm{i}\sigma\psi}\sqrt{\frac{2j+1}{4\pi}}\sqrt{\frac{(j-\mu)!(j+\mu)!}{(j-\sigma)!(j+\sigma)!}} \\ &\quad\times\frac{1}{2^\mu}(1+\cos\theta)^{\frac{\mu+\sigma}{2}}(1-\cos\theta)^{\frac{\mu-\sigma}{2}}P_{j-\mu}^{(\mu-\sigma,\mu+\sigma)}(\cos\theta)\,\mathrm{e}^{\mathrm{i}\mu\phi}. \end{aligned} \tag{A.24}$$

For other cases, it is necessary to use alternate expressions based on the relations (Magnus *et al* 1966)

$$P_n^{(-l,\beta)}(x) = \frac{\binom{n+\beta}{l}}{\binom{n}{l}}\left(\frac{x-1}{2}\right)^l P_{n-l}^{(l,\beta)}(x), \qquad P_0^{(\alpha,\beta)}(x) = 1. \tag{A.25}$$

Note that with $\sigma = 0$ we recover the expression of the normalized spherical harmonics

$$\begin{aligned} {}_0Y_{jm}(\hat{\mathbf{r}}) = Y_{jm}(\hat{\mathbf{r}}) &= (-1)^m\sqrt{\frac{2j+1}{4\pi}}\sqrt{(j-m)!(j+m)!}\,\frac{1}{j!2^m}(\sin\theta)^m P_{j-m}^{(m,m)}(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi} \\ &= \sqrt{\frac{2j+1}{4\pi}}\sqrt{\frac{(j-m)!}{(j+m)!}}\,P_j^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi} \end{aligned} \tag{A.26}$$

since we have the following relation between associated Legendre polynomials and Jacobi polynomials:

$$P_{j-m}^{(m,m)}(z) = (-1)^m 2^m (1-z^2)^{-\frac{m}{2}} \frac{j!}{(j+m)!} P_j^m(z), \tag{A.27}$$

for $m > 0$. We also recall the symmetry formula

$$P_j^{-m}(z) = (-1)^m \frac{(j-m)!}{(j+m)!} P_j^m(z). \tag{A.28}$$

Our expression of spherical harmonics is rather standard, in agreement with (Arfken 1985, Weisstein 2006).[3]

A.5. Transformation laws

We consider here the transformation law of the spin spherical harmonics under the rotation group. From the relation

$$\mathcal{R}\mathcal{R}_{{}^t\mathcal{R}\hat{\mathbf{r}}} = \mathcal{R}_{\hat{\mathbf{r}}} \tag{A.29}$$

for any $\mathcal{R} \in SO(3)$, and from the homomorphism $\xi(\mathcal{R}\mathcal{R}') = \xi(\mathcal{R})\xi(\mathcal{R}')$ between $SO(3)$ and $SU(2)$, we deduce from definition (A.12) of the spin spherical harmonics the transformation law

$$\begin{aligned}
{}_\sigma Y_{j\mu}({}^t\mathcal{R}\cdot\hat{\mathbf{r}}) &= \sqrt{\frac{2j+1}{4\pi}} D^j_{\sigma\mu}(\xi^\dagger(\mathcal{R}_{{}^t\mathcal{R}\cdot\hat{\mathbf{r}}})) = \sqrt{\frac{2j+1}{4\pi}} D^j_{\sigma\mu}(\xi^\dagger({}^t\mathcal{R}\mathcal{R}_{\hat{\mathbf{r}}}))\\
&= \sqrt{\frac{2j+1}{4\pi}} D^j_{\sigma\mu}(\xi^\dagger(\mathcal{R}_{\hat{\mathbf{r}}})\xi(\mathcal{R})) = \sqrt{\frac{2j+1}{4\pi}} \sum_\nu D^j_{\sigma\nu}(\xi^\dagger(\mathcal{R}_{\hat{\mathbf{r}}}))D^j_{\nu\mu}(\xi(\mathcal{R}))\\
&= \sum_\nu {}_\sigma Y_{j\nu}(\hat{\mathbf{r}}) D^j_{\nu\mu}(\xi(\mathcal{R})),
\end{aligned} \tag{A.30}$$

as expected if we think the special case ($\sigma = 0$) of the spherical harmonics.

Given a function $f(x)$ on the sphere S^2 belonging to the $(2j+1)$-dimensional Hilbert space $\mathcal{H}^{\sigma j}$ and a rotation $\mathcal{R} \in SO(3)$, we define the rotation operator $\mathcal{D}^{\sigma j}(\mathcal{R})$ for that representation by

$$(\mathcal{D}^{\sigma j}(\mathcal{R})f)(x) = f(\mathcal{R}^{-1}\cdot x) = f({}^t\mathcal{R}\cdot x). \tag{A.31}$$

Thus, in particular,

$$(\mathcal{D}^{\sigma j}(\mathcal{R})\, {}_\sigma Y_{j\mu})(\hat{\mathbf{r}}) = {}_\sigma Y_{j\mu}({}^t\mathcal{R}\cdot\hat{\mathbf{r}}). \tag{A.32}$$

The generators of the three rotations $\mathcal{R}^{(a)}$, $a = 1, 2, 3$, around the three usual axes, are the angular momentum operator in the representation. When $\sigma = 0$, we recover the usual SHs, and these generators are the usual angular momentum operators J^i (short notation for $J_i^{(j)}$) for that representation. In the general case $\sigma \neq 0$, we call them $\Lambda_a^{(\sigma j)}$. We study their properties below.

[3] Sometimes (e.g., (Arfken 1985)), the Condon–Shortley phase $(-1)^m$ is prepended to the definition of the spherical harmonics. Talman adopted this convention.

A.6. Infinitesimal transformation laws

Recalling that the components $J_a = -\mathrm{i}\epsilon_{abc}x^b\partial_c$ of the ordinary angular momentum operator are given in spherical coordinates by

$$\begin{aligned} J_3 &= -\mathrm{i}\partial_\phi, \\ J_+ &= J_1 + \mathrm{i}J_2 = \mathrm{e}^{\mathrm{i}\phi}(\partial_\theta + \mathrm{i}\cot\theta\,\partial_\phi), \\ J_- &= J_1 - \mathrm{i}J_2 = -\mathrm{e}^{-\mathrm{i}\phi}(\partial_\theta - \mathrm{i}\cot\theta\,\partial_\phi). \end{aligned} \tag{A.33}$$

We have introduced the 'spin' angular momentum operators:

$$\Lambda_3^{\sigma j} = J_3 = -\mathrm{i}\partial_\phi, \tag{A.34}$$

$$\Lambda_+^{\sigma j} = \Lambda_1^{\sigma j} + \mathrm{i}\Lambda_2^{\sigma j} = J_+ + \sigma\csc\theta\,\mathrm{e}^{\mathrm{i}\phi}, \tag{A.35}$$

$$\Lambda_-^{\sigma j} = \Lambda_1^{\sigma j} - \mathrm{i}\Lambda_2^{\sigma j} = J_- + \sigma\csc\theta\,\mathrm{e}^{-\mathrm{i}\phi}. \tag{A.36}$$

They obey the expected commutation rules,

$$\left[\Lambda_3^{\sigma j}, \Lambda_\pm^{\sigma j}\right] = \pm\Lambda_\pm^{\sigma j}, \qquad \left[\Lambda_+^{\sigma j}, \Lambda_-^{\sigma j}\right] = 2\Lambda_3^{\sigma j}. \tag{A.37}$$

These operators are the infinitesimal generators of the action of $SU(2)$ on the spin spherical harmonics:

$$\Lambda_3^{\sigma j}\,{}_\sigma Y_{j\mu} = \mu_\sigma Y_{j\mu} \tag{A.38}$$

$$\Lambda_+^{\sigma j}\,{}_\sigma Y_{j\mu} = \sqrt{(j-\mu)(j+\mu+1)}\,{}_\sigma Y_{j\mu+1} \tag{A.39}$$

$$\Lambda_-^{\sigma j}\,{}_\sigma Y_{j\mu} = \sqrt{(j+\mu)(j-\mu+1)}\,{}_\sigma Y_{j\mu-1}. \tag{A.40}$$

A.7. Integrals and $3j$-symbols

Specifying equation (A.8) to the spin spherical harmonics leads to the following orthogonality relations which are valid for j integer (and consequently σ integer):

$$\int_{S^2} {}_\sigma Y_{j\mu}(\hat{\mathbf{r}})({}_\sigma Y_{j'\nu}(\hat{\mathbf{r}}))^*\mu(\mathrm{d}\hat{\mathbf{r}}) = \delta_{jj'}\delta_{\mu\nu}. \tag{A.41}$$

We recall that in the integer case, the range of values assumed by the angle ϕ is $0 \leqslant \phi < 2\pi$. Now, if we consider half-integer j (and consequently σ), the range of values assumed by the angle ϕ becomes $0 \leqslant \phi < 4\pi$. The integral above has to be carried out on the 'doubled' sphere $\widetilde{S}^2$, and an extra normalization factor equal to $\frac{1}{\sqrt{2}}$ is needed in the expression of the spin spherical harmonics.

For a given integer σ, the set $\{{}_\sigma Y_{j\mu}, -\infty \leqslant \mu \leqslant \infty, j \geqslant \max(0, \sigma, m)\}$ forms an orthonormal basis of the Hilbert space $L^2(S^2)$. Indeed, at μ fixed so that $\mu \pm \sigma \geqslant 0$, the set

$$\left\{\sqrt{\frac{2j+1}{4\pi}}\sqrt{\frac{(j-\mu)!(j+\mu)!}{(j-\sigma)!(j+\sigma)!}}\frac{1}{2^\mu}(1+\cos\theta)^{\frac{\mu+\sigma}{2}}(1-\cos\theta)^{\frac{\mu-\sigma}{2}}P_{j-\mu}^{(\mu-\sigma,\mu+\sigma)}(\cos\theta),\ j \geqslant \mu\right\}$$

is an orthonormal basis of the Hilbert space $L^2([-\pi,\pi], \sin\theta\,\mathrm{d}\theta)$. The same holds for other ranges of values of μ by using alternate expressions like (A.25) for Jacobi polynomials. Then it suffices to view $L^2(S^2)$ as the tensor product $L^2([-\pi,\pi], \sin\theta\,\mathrm{d}\theta)\bigotimes L^2(S^1)$. Similar reasoning is valid for half-integer σ. Then, the Hilbert space to be considered is the space of 'fermionic' functions on the doubled sphere $\widetilde{S}^2$, i.e. such that $f(\theta, \phi + 2\pi) = -f(\theta, \phi)$.

Specifying equation (A.9) to the spin spherical harmonics leads to

$$
{}_{\sigma}Y_{j\mu}(\hat{\mathbf{r}})_{\sigma'}Y_{j'\mu'}(\hat{\mathbf{r}}) = \sum_{j''\mu''\sigma''} \sqrt{\frac{(2j+1)(2j'+1)(2j''+1)}{4\pi}} \\
\times \begin{pmatrix} j & j' & j'' \\ \mu & \mu' & \mu'' \end{pmatrix} \begin{pmatrix} j & j' & j'' \\ \sigma & \sigma' & \sigma'' \end{pmatrix} ({}_{\sigma''}Y_{j''\mu''}(\hat{\mathbf{r}}))^{*}. \tag{A.42}
$$

We easily deduce from (A.42) the following integral involving the product of three spherical spin harmonics (in the integer case, but an analog formula exists in the half-integer case) and with the constraint that $\sigma + \sigma' + \sigma'' = 0$:

$$
\int_{S^2} {}_{\sigma}Y_{j\mu}(\hat{\mathbf{r}})_{\sigma'}Y_{j'\mu'}(\hat{\mathbf{r}})_{\sigma''}Y_{j''\mu''}(\hat{\mathbf{r}})\mu(\mathrm{d}\hat{\mathbf{r}}) \\
= \sqrt{\frac{(2j+1)(2j'+1)(2j''+1)}{4\pi}} \begin{pmatrix} j & j' & j'' \\ \mu & \mu' & \mu'' \end{pmatrix} \begin{pmatrix} j & j' & j'' \\ \sigma & \sigma' & \sigma'' \end{pmatrix}. \tag{A.43}
$$

Note that this formula is independent of the presence of a constant phase factor of the type $\mathrm{e}^{\mathrm{i}\sigma\psi}$ in the definition of the spin spherical harmonics because of the *a priori* constraint $\sigma + \sigma' + \sigma'' = 0$. On the other hand, we have to be careful in applying equation (A.43) because of this constraint, i.e. since it has been derived from equation (A.42) on the ground that σ'' was already *fixed* at the value $\sigma'' = -\sigma - \sigma'$. Therefore, the computation of

$$
\int_{S^2} {}_{\sigma}Y_{j\mu}(\hat{\mathbf{r}})_{\sigma'}Y_{j'\mu'}(\hat{\mathbf{r}})_{\sigma''}Y_{j''\mu''}(\hat{\mathbf{r}})\mu(\mathrm{d}\hat{\mathbf{r}})
$$

for an arbitrary triplet $(\sigma, \sigma', \sigma'')$ should be carried out independently.

A.8. Important particular case: $j = 1$

In the particular case $j = 1$, we get the following expressions for the spin spherical harmonics:

$$
{}_{\sigma}Y_{10}(\hat{\mathbf{r}}) = \mathrm{e}^{\mathrm{i}\sigma\psi} \sqrt{\frac{3}{4\pi}} \frac{1}{\sqrt{(1+\sigma)!(1-\sigma)!}} \left(\cot\frac{\theta}{2}\right)^{\sigma} \cos\theta, \tag{A.44}
$$

$$
{}_{\sigma}Y_{11}(\hat{\mathbf{r}}) = -\mathrm{e}^{\mathrm{i}\sigma\psi} \sqrt{\frac{3}{4\pi}} \frac{1}{\sqrt{2(1+\sigma)!(1-\sigma)!}} \left(\cot\frac{\theta}{2}\right)^{\sigma} \sin\theta\, \mathrm{e}^{\mathrm{i}\phi}, \tag{A.45}
$$

$$
{}_{\sigma}Y_{1-1}(\hat{\mathbf{r}}) = (-1)^{\sigma}\, \mathrm{e}^{-\mathrm{i}\sigma\psi} \sqrt{\frac{3}{4\pi}} \frac{1}{\sqrt{2(1+\sigma)!(1-\sigma)!}} \left(\tan\frac{\theta}{2}\right)^{\sigma} \sin\theta\, \mathrm{e}^{-\mathrm{i}\phi}. \tag{A.46}
$$

For $\sigma = 0$, we recover a familiar formula connecting spherical harmonics to components of vector on the unit sphere:

$$
Y_{10}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}} z, \tag{A.47}
$$

$$
Y_{11}(\hat{\mathbf{r}}) = -\sqrt{\frac{3}{4\pi}} \frac{1}{\sqrt{2}} \sin\theta\, \mathrm{e}^{\mathrm{i}\phi} = -\sqrt{\frac{3}{4\pi}} \frac{x + \mathrm{i}y}{\sqrt{2}}, \tag{A.48}
$$

$$
Y_{1-1}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}} \frac{1}{\sqrt{2}} \sin\theta\, \mathrm{e}^{-\mathrm{i}\phi} = \sqrt{\frac{3}{4\pi}} \frac{x - \mathrm{i}y}{\sqrt{2}}. \tag{A.49}
$$

A.9. Another important case: $\sigma = j$

For $\sigma = j$, due to relations (A.25), the spin spherical harmonics reduce to their simplest expressions:

$$ {}_jY_{j\mu}(\hat{\mathbf{r}}) = (-1)^j\,\mathrm{e}^{\mathrm{i}j\psi}\sqrt{\frac{2j+1}{4\pi}}\sqrt{\binom{2j}{j+\mu}}\left(\cos\frac{\theta}{2}\right)^{j+\mu}\left(\sin\frac{\theta}{2}\right)^{j-\mu}\mathrm{e}^{\mathrm{i}\mu\phi}. \tag{A.50} $$

They are precisely the states which appear in the construction of the Gilmore–Radcliffe coherent states. Otherwise said, the latter and related quantization are just particular cases of our approach.

A.10. Spin coherent states

For a given pair (j,σ), we define the family of coherent states in the $(2j+1)$-dimensional Hilbert space $\mathcal{H}_{\sigma j}$:

$$ |x\rangle = |\theta,\phi\rangle = \frac{1}{\sqrt{\mathcal{N}(x)}}\sum_{\mu=-j}^{j} {}_\sigma Y^*_{j\mu}(x)|\sigma j\mu\rangle, \qquad |x\rangle \in \mathcal{H}_{\sigma j}, \tag{A.51} $$

with

$$ \mathcal{N}(x) = \sum_{\mu=-j}^{j} |{}_\sigma Y_{j\mu}(x)|^2 = \frac{2j+1}{4\pi}. $$

For $\sigma = j$, these coherent states identify the so-called *spin* or *atomic* or *Bloch* coherent states (Perelomov 1986). But for a given j and two different $\sigma \neq \sigma'$, the corresponding families are distinct because they live in different Hilbert spaces of the same dimension $2j+1$. This is due to the fact that the map between the two orthonormal sets is not unitary, since we should deal with expansions like

$$ {}_\sigma Y_{j\mu} = \sum_{j'\mu'} \mathcal{M}_{j'\mu',j\mu}(\sigma',\sigma)\,{}_{\sigma'}Y_{j'\mu'}, \tag{A.52} $$

where

$$ \mathcal{M}_{j'\mu',j\mu}(\sigma',\sigma) = \int_{S^2} ({}_{\sigma'}Y_{j'\mu'}(\hat{\mathbf{r}}))^*\,{}_\sigma Y_{j\mu}(\hat{\mathbf{r}})\mu(\mathrm{d}\hat{\mathbf{r}}) = [j'j\sigma'\sigma\mu]\delta_{\mu\mu'}. \tag{A.53} $$

The (non-trivial!) coefficient $[j'j\sigma\sigma'\mu]$ forces the sum to run on values of j' different to j.

A.11. Covariance properties of spin CS

The definition of the rotation operator $\mathcal{D}^{\sigma j}(\mathcal{R})$ was given in (A.31). Starting from a CS $|x\rangle$, let us consider the coherent state with the rotated parameter $\mathcal{R}\times x$. Due to the transformation property (A.30), the invariance of $\mathcal{N}(x)$ and the unitarity of $\mathcal{D}^j$, we find

$$ \begin{aligned} |\mathcal{R}\cdot x\rangle &= \frac{1}{\sqrt{\mathcal{N}(x)}}\sum_{\mu=-j}^{j} {}_\sigma Y^*_{j\mu}({}^t\mathcal{R}\cdot x)|\sigma j\mu\rangle \\ &= \frac{1}{\sqrt{\mathcal{N}(x)}}\sum_{\mu,\mu'=-j}^{j} {}_\sigma Y^*_{j\mu'}(x)(D^j_{\mu'\mu}(\xi(\mathcal{R}^{-1})))^\star|\sigma j\mu\rangle \\ &= \frac{1}{\sqrt{\mathcal{N}(x)}}\sum_{\mu'=-j}^{j} {}_\sigma Y^*_{j\mu'}(x)\sum_{\mu=-j}^{j} D^j_{\mu\mu'}(\xi(\mathcal{R}))|\sigma j\mu\rangle \\ &= \mathcal{D}^{\sigma j}(\mathcal{R})|x\rangle, \end{aligned} \tag{A.54} $$

where $\mathcal{D}^{\sigma j}$ have been defined in (A.31).

Hence, we get the (standard) covariance property of the spin CS:

$$\mathcal{D}^{\sigma j}(\mathcal{R})|\mathcal{R}^{-1}\cdot x\rangle = |x\rangle. \tag{A.55}$$

Note that this proves the equality between our coherent states and the $SU(2)$ Perelomov ones (Perelomov 1986).

A.12. Spin CS quantization

A classical observable on X is a function $f : X \mapsto \mathbb{C}$. To any such function f, we associate the operator A_f in $\mathcal{H}_{\sigma j}$ through the map:

$$f \mapsto A_f \equiv \int_X f(x)|x\rangle\langle x|\mathcal{N}(x)\mu(\mathrm{d}x). \tag{A.56}$$

Occasionally, we might use the notation $\tilde{f}$ for A_f.

In terms of its matrix elements in the basis of spin harmonics, this operator reads

$$\begin{aligned} A_f &= \sum_{\mu,\mu'=-j}^{j} \int_X f(x)\,{}_\sigma Y^*_{j\mu}(x)\,{}_\sigma Y_{j\mu'}(x)|\sigma j\mu\rangle\langle\sigma j\mu'|\mu(\mathrm{d}x) \\ &\equiv \sum_{\mu,\mu'=-j}^{j} [A_f]_{\mu\mu'}|\sigma j\mu\rangle\langle\sigma j\mu'|. \end{aligned} \tag{A.57}$$

A.13. Spin CS quantization of spin spherical harmonics

The quantization of an arbitrary spin harmonics ${}_\nu Y_{kn}$ yields an operator in $\mathcal{H}^{\sigma j}$ whose $(2j+1)\times(2j+1)$ matrix elements are given by the following integral resulting from (A.57):

$$\begin{aligned} [{}_\nu\widetilde{Y}_{kn}]_{\mu\mu'} &= \int_X {}_\sigma Y^*_{j\mu}(x)\,{}_\sigma Y_{j\mu'}(x)\,{}_\nu Y_{kn}(x)\mu(\mathrm{d}x) \\ &= \int_X (-1)^{\sigma-\mu}\,{}_{-\sigma}Y_{j-\mu}(x)\,{}_\sigma Y_{j\mu'}(x)\,{}_\nu Y_{kn}(x)\mu(\mathrm{d}x). \end{aligned} \tag{A.58}$$

As asserted above, it is only when $\nu-\sigma+\sigma=0$, i.e. when $\nu=0$, that the integral (A.58) is given in terms of a product of two $3j$-symbols as follows:

$$\begin{aligned} [\widetilde{Y}_{kn}]_{\mu\mu'} &= \int_X {}_\sigma Y^*_{j\mu}(x)\,{}_\sigma Y_{j\mu'}(x)Y_{kn}(x)\mu(\mathrm{d}x) \\ &= \int_X (-1)^{\sigma-\mu}\,{}_{-\sigma}Y_{j-\mu}(x)\,{}_\sigma Y_{j\mu'}(x)Y_{kn}(x)\mu(\mathrm{d}x) \\ &= (-1)^{\sigma-\mu}(2j+1)\sqrt{\frac{(2k+1)}{4\pi}}\begin{pmatrix} j & j & k \\ -\mu & \mu' & n \end{pmatrix}\begin{pmatrix} j & j & k \\ -\sigma & \sigma & 0 \end{pmatrix}. \end{aligned} \tag{A.59}$$

A.14. Checking quantization in the simplest case: $j = 1$

With the notations of the text, we find for the matrix elements of the CS quantized versions of the above spherical harmonics,

$$[\widetilde{Y}_{10}]_{mn} = \sigma\sqrt{\frac{3}{4\pi}}\frac{1}{j(j+1)}m\delta_{mn}, \tag{A.60}$$

$$[\widetilde{Y}_{11}]_{mn} = -\sigma\sqrt{\frac{3}{4\pi}}\frac{1}{j(j+1)}\sqrt{\frac{(j-n)(j+n+1)}{2}}\delta_{mn+1}, \tag{A.61}$$

$$[\widetilde{Y}_{1-1}]_{mn} = \sigma\sqrt{\frac{3}{4\pi}}\frac{1}{j(j+1)}\sqrt{\frac{(j+n)(j-n+1)}{2}}\delta_{mn-1}. \tag{A.62}$$

Comparing with the actions (A.38)–(A.40) of the spin angular momentum on the spin-σ spherical harmonics, we have the identification

$$\widetilde{Y}_{10} = \sigma\sqrt{\frac{3}{4\pi}}\frac{1}{j(j+1)}\Lambda_3, \tag{A.63}$$

$$\widetilde{Y}_{11} = -\sigma\sqrt{\frac{3}{8\pi}}\frac{1}{j(j+1)}\Lambda_+, \tag{A.64}$$

$$\widetilde{Y}_{1-1} = \sigma\sqrt{\frac{3}{8\pi}}\frac{1}{j(j+1)}\Lambda_-. \tag{A.65}$$

Hence, we can conclude on the following identification between quantized versions of the components of the vector on the unit sphere and the components of the spin angular momentum operator:

$$\widetilde{x} = \frac{\sigma}{j(j+1)}\Lambda_1, \tag{A.66}$$

$$\widetilde{y} = \frac{\sigma}{j(j+1)}\Lambda_2, \tag{A.67}$$

$$\widetilde{z} = \frac{\sigma}{j(j+1)}\Lambda_3. \tag{A.68}$$

A.15. Rotational covariance properties of operators

By construction, the operators $\widetilde{{}_\nu Y_{kn}}$ acting on $\mathcal{H}^{\sigma j}$ are tensorial irreducible. Indeed, under the action of the representation operator $\mathcal{D}^{\sigma j}(\mathcal{R})$ in $\mathcal{H}^{\sigma j}$, due to (A.55), the rotational invariance of the measure and $\mathcal{N}(x)$, and (A.30), they transform as

$$\begin{aligned}
\mathcal{D}^{\sigma j}(\mathcal{R})\widetilde{{}_\nu Y_{kn}}\mathcal{D}^{j}(\mathcal{R}^{-1}) &= \int_X {}_\nu Y_{kn}(x)|\mathcal{R}\cdot x\rangle\langle \mathcal{R}\cdot x|\mathcal{N}(x)\mu(\mathrm{d}x) \\
&= \int_X {}_\nu Y_{kn}(\mathcal{R}^{-1}\cdot x)|x\rangle\langle x|\mathcal{N}(x)\mu(\mathrm{d}x) \\
&= \sum_{n'} D^k_{n'n}(\xi(\mathcal{R}))\int_X {}_\nu Y_{kn'}(x)|x\rangle\langle x|\mathcal{N}(x)\mu(\mathrm{d}x) \\
&= \sum_{n'} \widetilde{{}_\nu Y_{kn'}}D^k_{n'n}(\xi(\mathcal{R})).
\end{aligned} \tag{A.69}$$

Therefore, the Wigner–Eckart theorem (Edmonds 1968) tells us that the matrix elements of the operator $\widetilde{{}_\nu Y_{kn}}$ with respect to the SSH basis $\{{}_\sigma\widetilde{Y}_{jm}\}$ are given by

$$[{}_\nu\widetilde{Y}_{kn}]_{mm'} = (-1)^{j-m}\begin{pmatrix} j & j & k \\ -m & m' & n\end{pmatrix}\mathcal{K}(\nu,\sigma,j,k). \tag{A.70}$$

Note that the presence of the $3j$-symbol in (A.70) implies the selection rules $n+m'=m$ and the triangular rule $0 \leqslant k \leqslant 2j$. The proportionality coefficient $\mathcal{K}$ can be computed directly from (A.58) by choosing therein suitable values of m, m'.

On the other hand, we have by definition (A.30), (A.32)

$$\sum_{n'} {}_\nu Y_{kn'}D^k_{n'n}(\xi(\mathcal{R})) = \mathcal{D}^{\nu k}(\mathcal{R})_\nu Y_{kn}.$$

Thus, from the formula above,

$$\mathcal{D}^{\sigma j}(\mathcal{R})_\nu \widetilde{Y_{kn}} \mathcal{D}^{j}(\mathcal{R}^{-1}) = \mathcal{D}^{\nu k}\widetilde{(\mathcal{R})_\nu Y_{kn}}.$$

In the special case $\nu = 0$,

$$\mathcal{D}^{\sigma j}(\mathcal{R}) \widetilde{Y_{kn}} \mathcal{D}^{j}(\mathcal{R}^{-1}) = \mathcal{D}^{0k}\widetilde{(\mathcal{R}) Y_{kn}}. \tag{A.71}$$

Its infinitesimal version for each one of the three rotations $\mathcal{R}_i$ reads as

$$\left[\Lambda_i^{(\sigma j)}, \widetilde{Y_{kn}}\right] = \widetilde{J_i^{(k)} Y_{kn}}. \tag{A.72}$$

Appendix B. Symmetrization of the commutator

We want to prove that

$$S\left(\left[J_3, J_1^{\alpha_1} J_2^{\alpha_2} J_3^{\alpha_3}\right]\right) = \left[J_3, S\left(J_1^{\alpha_1} J_2^{\alpha_2} J_3^{\alpha_3}\right)\right],$$

where J_i is a representation of $so(3)$.

Let us make a first comment on the symmetrization

$$S\left(J_1^{\alpha_1} J_2^{\alpha_2} J_3^{\alpha_3}\right) = \frac{1}{l!}\sum_{\sigma\in S_l} J_{i_{\sigma(1)}} \dots J_{i_{\sigma(l)}},$$

where $l = \alpha_1 + \alpha_2 + \alpha_3$. The terms of the sum are not all distinct, since the exchange of, say, two J_1 gives the same term: each term appears in fact $\alpha_1!\alpha_2!\alpha_3!$ times, so that there are $l!/(\alpha_1!\alpha_2!\alpha_3!)$ distinct terms. This is the number of sequences of length l, with values in $\{1, 2, 3\}$, where there are α_i occurrences of the value i (for $i = 1, 2, 3$). One denotes this set as $U_{\alpha_1,\alpha_2,\alpha_3}$. After grouping of identical terms, one obtains

$$S\left(J_1^{\alpha_1} J_2^{\alpha_2} J_3^{\alpha_3}\right) = \frac{\alpha_1!\alpha_2!\alpha_3!}{l!}\sum_{u\in U_{\alpha_1,\alpha_2,\alpha_3}} J_{u_1} \dots J_{u_l},$$

where all the terms of the summation are now different.

Let us now calculate $S\left(\left[J_3, J_1^{\alpha_1} J_2^{\alpha_2} J_3^{\alpha_3}\right]\right)$. First, we write

$$\left[J_3, J_1^{\alpha_1} J_2^{\alpha_2} J_3^{\alpha_3}\right] = \underbrace{\left[J_3, J_1^{\alpha_1}\right] J_2^{\alpha_2} J_3^{\alpha_3}}_{A} + \underbrace{J_1^{\alpha_1}\left[J_3, J_2^{\alpha_2}\right] J_3^{\alpha_3}}_{B},$$

with

$$A = \sum_{k=1}^{\alpha_1} \underbrace{J_1 \dots J_1}_{k-1 \text{ terms}} J_2 \underbrace{J_1 \dots J_1}_{\alpha_1 - k \text{ terms}} J_2^{\alpha_2} J_3^{\alpha_3}.$$

The different terms in A give the same symmetrized. Thus,

$$S(A) = \alpha_1 S\left(J_1^{\alpha_1-1} J_2^{\alpha_2+1} J_3^{\alpha_3}\right) \tag{B.1}$$

$$= \alpha_1 \frac{(\alpha_1-1)!(\alpha_2+1)!\alpha_3!}{l!} \sum_{u\in U_{\alpha_1-1,\alpha_2+1,\alpha_3}} J_{u_1} \dots J_{u_l}. \tag{B.2}$$

Similarly, for B,

$$S(B) = -\alpha_2 \frac{(\alpha_1+1)!(\alpha_2-1)!\alpha_3!}{l!} \sum_{u\in U_{\alpha_1+1,\alpha_2-1,\alpha_3}} J_{u_1} \dots J_{u_l}.$$

Now we calculate

$$\begin{aligned} I &= \left[J_3, S\left(J_1^{\alpha_1} J_2^{\alpha_2} J_3^{\alpha_3}\right)\right] \\ &= \frac{\alpha_1!\alpha_2!\alpha_3!}{l!} \sum_{u\in U_{\alpha_1,\alpha_2,\alpha_3}} \sum_{k=1}^{l} J_{u_1}\ldots J_{u_{k-1}}\left[J_3, J_{u_k}\right] J_{u_{k+1}}\ldots J_{u_l}. \end{aligned}$$

The sum splits in two parts, according to the value of $u_k = 1$ or 2

$$I = A' + B',$$

with

$$A' = \frac{\alpha_1!\alpha_2!\alpha_3!}{l!} \sum_{u\in U_{\alpha_1,\alpha_2,\alpha_3}} \sum_{k|u_k=1} J_{u_1}\ldots J_{u_{k-1}} J_2 J_{u_{k+1}}\ldots J_{u_l}$$

and

$$B' = -\frac{\alpha_1!\alpha_2!\alpha_3!}{l!} \sum_{u\in U_{\alpha_1,\alpha_2,\alpha_3}} \sum_{k|u_k=2} J_{u_1}\ldots J_{u_{k-1}} J_1 J_{u_{k+1}}\ldots J_{u_l}.$$

Let us examine the constituents of A'. They are of the form $J_{u_1}\ldots J_{u_l}$ with $u \in U_{\alpha_1-1,\alpha_2+1,\alpha_3}$. Their number is $l!/(\alpha_1!\alpha_2!\alpha_3!) \times \alpha_1$, but they are not all different. Each monomial is issued from a term where J_1 has been transformed into J_2. Since there are $\alpha_2 + 1$ occurrences of J_2 in each term, each monomial appears $\alpha_2 + 1$ times. We now group these identical terms

$$A' = \frac{\alpha_1!\alpha_2!\alpha_3!}{l!}(\alpha_2 + 1) \sum_{?} J_{u_1}\ldots J_{u_l}.$$

It remains to determine the definition set of the summation. Let us first estimate the number of its terms, namely

$$N = \frac{l!}{\alpha_1!\alpha_2!\alpha_3!}\frac{\alpha_1}{\alpha_2 + 1} = \frac{l!}{(\alpha_1 - 1)!(\alpha_2 + 1)!\alpha_3!}.$$

This is the number of elements in $U_{\alpha_1-1,\alpha_2+1,\alpha_3}$. On the other hand, all the elements of $U_{\alpha_1-1,\alpha_2+1,\alpha_3}$ appear. In the contrary case, the retransformation of J_2 into J_1 would provide some elements not appearing in I, which cannot be. There results that the sum comprises exactly all symmetrized of $J_1^{\alpha_1-1} J_2^{\alpha_2+1} J_3^{\alpha_3}$. Thus,

$$\begin{aligned} A' &= \frac{\alpha_1!\alpha_2!\alpha_3!}{l!}(\alpha_2 + 1) \sum_{u\in U_{\alpha_1-1,\alpha_2+1,\alpha_3}} J_{u_1}\ldots J_{u_l} \\ &= \alpha_1 \frac{(\alpha_1 - 1)!(\alpha_2 + 1)!\alpha_3!}{l!} \sum_{u\in U_{\alpha_1-1,\alpha_2+1,\alpha_3}} J_{u_1}\ldots J_{u_l} \\ &= S(A). \end{aligned}$$

The application of the same treatment to B' leads to the proof.

References

Ali S T, Antoine J-P and Gazeau J-P 1993 Continuous frames in Hilbert spaces *Ann. Phys.* **222** 1–37

Ali S T, Antoine J-P and Gazeau J-P 2000 *Coherent States, Wavelets and their Generalizations (Graduate Texts in Contemporary Physics)* (New York: Springer)

Ali S T, Engliš M and Gazeau J-P 2004 Vector coherent states from Plancherels theorem, Clifford algebras and matrix domains *J. Phys. A: Math. Gen.* **37** 6067–90

Arfken G 1985 Spherical harmonics and integrals of the products of three spherical harmonics 12.6 and 12.9 *Mathematical Methods for Physicists* 3rd edn (Orlando, FL: Academic) pp 680–5 and 698–700
Berezin F A 1975 General concept of quantization *Commun. Math. Phys.* **40** 153–74
Campbell W B 1971 Tensor and spinor spherical harmonics and the spin-s harmonics ${}_sY_{lm}(\theta, \phi)$ *J. Math. Phys.* **12** 1763–70
Daubechies I 1992 *Ten Lectures on Wavelets* CBMS-NSF, Regional conference series in applied mathematics vol 61 (Philadelphia: SIAM)
Edmonds A R 1968 *Angular Momentum in Quantum Mechanics* 2nd edn (Princeton, NJ: Princeton University Press) (revised printing)
Freidel L and Krasnov K 2002 The fuzzy sphere $*$-product and spin networks *J. Math. Phys.* **43** 1737–54
Garidi T, Gazeau J-P, Huguet E, Lachièze-Rey M and Renaud J 2003 Quantization of the sphere with coherent states *Int. J. Theor. Phys.* **42** 1301–10
Lachièze-Rey M, Gazeau J-P, Huguet E, Renaud J and Garidi T 2003 Quantization of the sphere with coherent states *Int. J. Theor. Phys.* **42** 1301–10
Gazeau J-P, Garidi T, Huguet E, Lachièze-Rey M and Renaud J 2004 Examples of Berezin–Toeplitz Quantization: Finite Sets and Unit Interval *Proc. CRM and Lecture Notes* **34** 67–76
Gazeau J-P and Piechocki W 2004 Coherent states quantization of a particle in de Sitter space *J. Phys. A: Math. Gen.* **37** 6977–86
Gazeau J P, Mourad J and Quéva J 2006 Fuzzy de Sitter space-times via coherent state quantization *Proc. 26th ICGTMP (New York)* ed J Birman and S Catto (New York: Springer) to appear (*Preprint* quant-ph/0610222)
Gilmore R 1972 Geometry of symmetrized states *Ann. Phys., NY* **74** 391–463
Gilmore R 1974 On properties of coherent states *Rev. Mex. Fis.* **23** 143–87
Goldberg J N, Macfarlane A J, Newman E T, Rohrlich F and Sudarshan E C G 1967 Spin-s spherical harmonics and ∂ *J. Math. Phys.* **8** 2155–61
Grosse H and Prešnajder P 1993 The construction of non-commutative manifolds using coherent states *Lett. Math. Phys.* **28** 239–50
Hall B and Mitchell J J 2002 Coherent states on spheres *J. Math. Phys.* **43** 1211–36 (*Preprint* quant-ph/0109086)
Hu W and White M 1997 CMB anisotropies: total angular momentum method *Phys. Rev.* D **56** 596–615
Klauder J R 1963a Continuous-representation theory: I. Postulates of continuous-representation theory *J. Math. Phys.* **4** 1055–8
Klauder J R 1963b Continuous-representation theory: II. Generalized relation between quantum and classical dynamics *J. Math. Phys.* **4** 1058–73
Klauder J R 1995 Quantization without quantization *Ann. Phys.* **237** 147–60
Klauder J R and Skagerstam B S 1985 *Coherent States—Applications in Physics and Mathematical Physics* (Singapore: World Scientific)
Kowalski K and Rembielinski J 2000 Quantum mechanics on a sphere and coherent states *J. Phys. A: Math. Gen.* **33** 6035–48
Kowalski K and Rembielinski J 2001 The Bargmann representation for the quantum mechanics on a sphere *J. Math. Phys.* **42** 4138–47
See Lieb E 1994 *Coherent States: Past, Present and Future* ed D H Feng, J R Klauder and M Strayer (Singapore: World Scientific)
Madore J 1995 *An Introduction to Noncommutative Differential Geometry and its Physical Applications* (Cambridge: Cambridge University Press)
Magnus W, Oberhettinger F and Soni R P 1966 *Formulas and Theorems for the Special Functions of Mathematical Physics* 3rd edn (Berlin: Springer)
Newman E T and Penrose R 1966 Note on the Bondi–Metzner–Sachs Group *J. Math. Phys.* **7** 863–70
Perelomov A M 1972 Coherent states for arbitrary Lie group *Commun. Math. Phys.* **26** 222–36
Perelomov A M 1986 *Generalized Coherent States and their Applications* (Berlin: Springer)
Radcliffe J M 1971 Some properties of spin coherent states *J. Phys. A: Math. Gen.* **4** 313–23
Talman J D 1968 *Special Functions, A Group Theoretical Approach* (New York: Benjamin)
Taylor W 2001 M(atrix) theory *Rev. Mod. Phys.* **73** 419–61
Weisstein E W 2006 Spherical Harmonic, from MathWorld—A Wolfram Web Resource. http://mathworld.wolfram.com/SphericalHarmonic.html